**ECE REFERENCES & SUPPLEMENTAL INFORMATION**

**Email: baroody@illinois.edu**

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**APPENDIX A: Some Potentially Useful General References**

Baroody, A. J., with Coslick, R. T. (1998). *Fostering children’s mathematical power: An investigative approach to K-8 mathematics instruction*. Mahwah, NJ: Lawrence Erlbaum Associates.

Baroody, A. J. (2008). Why children have difficulties mastering the basic number facts and how to help them. In J. M. Bay-Williams & K. Karp (Eds.), *Growing Professionally: Readings from NCTM Publications for Grades K-8* (pp. 284–294). Reston, VA: National Council of Teachers of Mathematics

Baroody, A. J., & Li, X. (2009). Mathematics instruction that makes sense for 2 to 5 year olds. In E. L. Essa & M. M. Burnham (Eds.), *Informing our practice: Useful research on young children’s development* (pp. 119–135). Washington, DC: The National Association for the Education of Young Children.

Frye, D., Baroody, A. J., Burchinal, M., Carver, S. M., Jordan, N. C., & McDowell, J. (2013). *Teaching math to young children: A practice guide*. Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education. <<http://ies.ed.gov/ncee/wwc/practiceguide.aspx?sid=18>>.

**APPENDIX B: REFERENCES ON LEARNING PROGRESSIONS**

Confrey, J., Nguyen, K.H., Lee, K., Panorkou, N., Corley, A.K., & Maloney, A P. (2012). *Turn-on common core math: Learning trajectories for the common core state standards for mathematics*. Retrieved from http:www.turnonccmath.net

Organizes the *CCSSM* for measurement standards sequentially and in relation to other standards, in measurement, number, and geometry (Confrey, Maloney, & Corley, 2014). Drawing on relevant research, proposes “bridging standards” to fill gaps in the *CCSSM* sequence.

Clements, Douglas H., and Julie Sarama. 2009. *Learning and teaching early math: The learning trajectories approach*. New York: Routledge. [ISBN: 9780415995917]

This volume is a part of the prestigious and selective Studies in Mathematical Thinking and Learning series. A target audience for this practical and important book is pre- and in-service teachers. The book underscores how learning trajectories can help teachers more effectively plan, implement, and assess learning.

Daro, Philip, Frederic A. Mosher, Tom Corcoran, and Jeffrey Barrett. 2011. *Learning trajectories in mathematics: A foundation for standard, curriculum, assessment, and instruction*. Philadelphia: Consortium for Policy Research in Education. [class:report]

This brief (79 pp.), but highly informative, reference work reviews and parses definitions of learning trajectories, describes their role and value in assessment and adaptive instruction, and explains their relation to standards (goals), particularly the Common Core State Standards.

Fuson, Karen C., William M. Carroll, and Jane V. Drueck. 2000. Achievement results for second and third graders using the standards-based curriculum Everyday Mathematics. *Journal for Research in Mathematics Education* 31.3: 277–295.

The authors argue that learning trajectories can help focus teachers’ attention on teaching students rather than merely implementing a curriculum. Put differently, successful instruction focuses on children’s learning—their progress through a conceptually based trajectory—and not on progressing through a curriculum. Available \*online[http://www.mendeley.com/research/achievement-results-second-third-graders-using-standards-based-curriculum-everyday-mathematics-1/]\* by subscription.

Sarama, Julie, and Douglas H. Clements. 2009. *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge. [ISBN: 9780805863086]

This volume is a part of the prestigious and selective Studies in Mathematical Thinking and Learning series. It provides an extensive and comprehensive review of research on the learning of mathematics from birth to the primary grades, which serves as the empirical basis for learning trajectories. Tables summarize trajectories for the key domains of early childhood mathematics education.

**APPENDIX C: REFERENCES ON BIG IDEAS**

Baroody, Arthur J., Yingying Feil, and Amanda R. Johnson. 2007. An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education* 38:115–131.

Contrary to Star 2005, although superficial procedural and conceptual knowledge may exist independently, deep procedural knowledge cannot exist without (relatively) deep conceptual knowledge or vice versa. In addition to connectedness, other important knowledge qualities include the degree of organization, abstractness (the generality or breadth of application), and accuracy. Big ideas are a key basis for integrating concepts and procedures. Illustrative example: equal patitioning.

Paulos, John Allen. 1991. *Beyond numeracy: Ruminations of a numbers man.* New York: Alfred A. Knopf. [ISBN: 9780394586403]

In a series of two- to six-page essays, mathematical concepts or topics such as “algebra,” “coincidence,” “game theory,” “probability,” and “zero” are clearly described. The entries are alphabetically arranged for easy access. Paulos remarks: “stress a few basic principles and [leave] most of the details to [the student]” (Paulos 1991, p. 7). In effect, if children understand the big ideas, most may be able to rediscover or reinvent many basic mathematical principles, properties, and procedures with minimal guidance.

**APPENDIX D: REFERENCES ON**

**LEARNING PROGRESSIONS**

**AND BIG IDEAS**

Baroody, Arthur J., Meng-lung Lai, and Kelly S. Mix. 2006. The development of young children’s number and operation sense and its implications for early childhood education. In *Handbook of research on the education of young children*. Edited by Bernard Spodek and Olivia N. Saracho, 187–221. Mahwah, NJ: Lawrence Erlbaum Associates. [ISBN: 9780805847208]

Part 1 lays out a research-based developmental trajectory, the basis of which is verbal number recognition (subitizing). Part 2 includes a recommendation for using (a) the Vygotskian approach to guide *how* to teach mathematics to young children; (b) big ideas as goals to guide *what* to teach and (c) research on learning trajectories and developmental readiness to guide *when* to teach.

Fosnot, Catherine Twomey, and Maarten Dolk. 2001. \*Young mathematicians at work[http://www.heinemann.com/products/E00353.aspx]\*. Portsmouth, NH: Heinemann.

Draws on the work of the Dutch mathematician Hans Freudenthal and research in urban schools. provides “a landscape for learning”—a map that illustrates the big ideas and progressive strategies applied to four- to eight-year-olds that are needed to construct a deep understanding of number, addition, and subtraction. Illustrates the use of rich problematic situations to promote autonomous inquiry, problem solving, and “mathematizing.” Provides worthwhile goals (big ideas) and trajectories for helping four- to eight-year-olds construct a deep understanding of key aspects of mathematics.

**APPENDIX E: EXAMPLES OF LEARNING PROGRESSIONS**

**E.1: Progression for Early Object Counting Development (developed for the SRI/NC Consortium)**

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| **SKILLS** | **CONCEPTS NEEDED** |
| **Subitizing** | **Concept of the Cardinal Numbers 1 to 3** |
| **Initial one-to-one counting effort**  With collections up to 5 scattered objects, makes some effort at one-for one counting but may slip up with the one-to-one process (e.g., have trouble simultaneously starting or stopping the verbal counting and pointing processes and label an object with more than one number word), fail to use the standard counting sequence, or fail to keep track (e.g., later recounts an object or misses an object). That is, the child no longer simply makes a “skim “error (simply waves a finger over the collection like a wand while spewing number words) or a “flurry” of errors (more than errors).  Possible errors (errors with three boxes can occur multiple times):  ***Sequence error***  ☐ (uses an incorrect counting word sequence)  ***Correspondence errors***  ☐ Says more than 1 number word for 1st object  ☐ Points to 1st object but does not say number word  ☐ Says more than 1 number word for a middle object  ☐ ☐ ☐ Points to middle object but does not say number word  ☐ ☐ ☐ Says more than 1 number word for the last object  ☐ Points to the lastobject but does not say number word  ***Keeping track errors***  ☐ ☐ ☐ Counts an object and later recounts it  ☐ ☐ ☐ Skips an item (fails to ever point to the item)  TOTAL BOXES CHECKED?: | Knows that counting involves pointing to objects in the collection and tagging each with a number word. However, the following counting principles or the knowledge to implement them effectively are only emerging:,  **1. Stable order principle:** the same unique number word tags must always be used in the same order.  **2.One-to-one principle:** each item must be tagged with one and only one number, which entails (a) assigning one counting word for each point and keeping track of which items have been counted and which need to be counted. |
| **Intermediate one-to-one counting**  Consistently counts scrambled collections of up to 5 objects using the correct number word sequence and always using one number word when pointing to an object but may still make one or two keeping-track errors.    Note that children at this level typically have no difficulty counting a collection of 5 objects that are in a line, because the demands of keeping track of which items have already been counted and which need to counted are minimal. | **Understand and effectively implements the stable order principle**  (can recite number words correctly for number 1 to 5)  **and understands the**  **one-to-one principle**  (one-to one-tagging for sets up to 5) but still learning keeping track strategies necessary to implement the principle effectively. |
| **Advanced one-to-one counting**  Consistently counts scrambled collections of up to 5 accurately and without any sequence, correspondence, or keeping-track errors  Note that ending with correct number word is not a guarantee this criterion has been fulfilled because, for example, two keeping track errors (skipping one object and doubling back to recount another object can cancel each other . | **Stable order principle:** number word tags must always be used in the same order  **One-to-one principle**:  (a) Can do one-to one-tagging for sets of 5 and fewer and (b)  can keep track of which items have already been counted and need to be counted for randomly arranged sets of 5 or fewer. |
| **Meaningful one-to-one counting (collections to 5)**  Knows the last number counted is the total quantity and that the value of a collection of objects does not change unless items are added or removed. | **Cardinality Principle:** (Knows the last number stated represents the total quantity) |
| **Advanced meaningful one-to-one counting**  Understands the stability of cardinality: can predict the outcome of a recount (a) with a different starting point will result in the same cardinal number, (b) after a collection has been visibly rearranged will have the same total; and (c) with an arithmetic transformation will result in a different number. | Recognizes that the order in which items are (correctly) counted does not affect the total (**Order-Irrelevance Principle**) and that changes in the arrangement of items does not affect the total (**Number Identity**) but that adding or taking away items does change the total. |
| **Counts out a specified number of (SetProduction)**  Consistently counts out sets of 4 to 6 objects from a larger set of 10 objects. | **Count-Cardinality Concept:** Understands that the number requested specifies when to stop the counting out process. |
| **Meaningful one-to-one counting (collections to 10)**  Applies counting to finding the total when one object is added to a set (6-10 objects).  Child uses counting accurately to determine the sum whether s/he re-counts the original set again and then counts once more for added object (1-2-3-4-5-6 --- 7; concrete counting-all); starts with the number in the original set and counts from there (6 – 7) or continues the count one time by counting the added item (7; concrete counting-on) ; or uses a more advanced strategy (without counting, automatically states the sum). | Can apply concrete counting–all or concrete counting-on strategies for adding one object. |

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| **E.2: A Learning Progression for Early Number Development** (Adapted and expanded from Frye et al., 2013) | |
| ***Cardinal concept of small numbers + small number recognition (subitizing)***  **(1 and 2 first, then 3, and—in time—4 to 6)** | By seeing different examples of a number labeled with a unique number word and non-examples labeled with other number words, children construct of precise cardinal concepts one, two, and three (Palmer & Baroody, 2011). For example, seeing various pairs (e.g., , , , ) labeled “two” can help a child recognize that this number word refers to number (as opposed to a particular shape, color, or other feature irrelevant to number) and multiple items at that (as opposed to a singular item), and non-examples (“take one cookie, not two”) can help a child understand that “two” refers only to pairs. These concepts permit ***small number recognition*:** The ability to immediately recognize and label small collections with an appropriate number word. |
| ***Ordinal concept of small numbers* (collections of 1 to about 3 items)** | ***Small number recognition*** enables children to see that “two is *more* than one” item and that “three is more than “two” items (understand the term “more” and that numbers have an ***ordinal meaning***. |
| ***Meaningful object counting***  **(including the count-cardinality principle)** | ***Small number recognition*** enables children to understand the principles underlying ***meaningful counting:*** *stable order, one-to-one,* and *cardinality principles*. For example, by watching an adult count a small collection a child can recognize as “three,” s/he can understand why the last number word in the count is emphasized or repeated—it represents the total or how many (the cardinal value of the collection). |
| ***Increasing magnitude principle* + *counting-based number comparisons* (especially collections larger than 3)** | ***Small number recognition*** and ***ordinal number concept*** permit discovery of the **increasing magnitudeprinciple:** the counting sequence represent increasingly larger quantities. This enables them to use ***meaningful object counting*** to determine the larger of two collections (e.g., 7 items is more than 6 items because you have to count further to get to *seven* than you do for *six*). |
| ***Number-after knowledge* of the counting sequence** | Familiarity with the counting sequence enables a child to enter the sequence at any point and ***specify the next number*** instead of always counting from one. |
| ***Mental comparisons of close/neighboring number***  **(number after = more)** | The use of the ***increasing magnitude principle*** and ***number-after knowledge*** enables children to determine efficiently and ***mentally compare even close numbers*** such as the larger of two neighboring numbers (e.g., “Which is more seven or eight?—eight”) |
| ***Successor Principle***  **(Number after = 1 more)** | **Small number recognition** enables children to see that “two” is exactly one more than “one” item and that “three” is exactly one more than “two” items, and this can help them understand the ***successor principle*** (each successive number in the counting sequence is exactly one more than the previous number). |
| ***Reconceptualization of the counting sequence as the (positive) integer sequence*** | The ***successor principle*** enables children to view the counting sequence as *n*, *n*+1, [*n*+1]+1, … (***positive integer sequence***)—a linear representation of number. |
| ***Informal mathematical induction*** | An understanding of the ***successor principle*** and a linear representation of the ***(positive) integer sequence*** provide a basis for informal mathematical induction. |
| ***Infinite succession principle (concept of infinity)*** | ***Informal mathematical induction*** enables children to use their number-after knowledge and the successor principle to realize that the positive integer sequence can, in principle, go on forever. |

Baroody, A. J. (in preparation). Reasoning and Sense Making in Grades PK-2: Using Number and Arithmetic Instruction as a Basis for Fostering Mathematical Reasoning. In M. T. Battista (Ed.), *Reasoning and Sense Making in the Elementary Grades.* Reston, VA: National Council of Teachers of Mathematics.

**E.3 Developmental Progression: From Number Sense to Fluency with Basic Combinations (Based on brief prepared for the National Governors’ Association)**

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| **Learns first few number words** | Learns number words such as “one,” “two,” and “three” but does not realize they represent a particular exact amount. |
| **Small-number recognition**  **(subitizing)** | By seeing many different examples of a number (e.g., two eyes, two hands, two socks, two shoes, two cars) labeled with the same number word and non-examples of the number labeled with other number words (e.g., three cars), children can construct precise concepts of one, two, and three and can recognize and immediately label small collections with the appropriate number. |
| ***Hiding game.*** |
| **Meaningful object counting** | Meaningful object counting requires two things: (a) counting in a one-to-one fashion (linking one and only one number word to each object in a collection) and (b) understanding the cardinality principle (the last number word used in the counting process indicates the total or cardinal number). |
| ***Animal Spots.*** |
| **Increasing magnitude concept + counting-based comparisons of collections larger than three** | Small-number recognition enables children toSEE that “two” is more than “one” item and that “three” is more than “two” items, and this can help them understand that the counting sequence represents increasingly larger collections or numbers. (e.g., seven items is more than six items because you have to count further to get to seven than you do to get to six). |
| ***Number Guess.***  ***Cards More Than.*** |
| **Number-after knowledge** | Familiarity with the counting sequence enables a child to enter the sequence at any point and specify the next number instead of always counting from one. |
| ***Dominoes After***. |
| **Comparisons of close or neighboring numbers** | Once children have number-after knowledge and can recognize that each number represents "one more," they can determine, efficiently and mentally, the larger of two adjacent or close numbers (e.g., seven and eight). |
| ***Chase Game.*** |
| **Number after equals one more**  **a number and one more is the number after** | Small number recognition enables children to see that “two” is one more than “one” items and that “three” is one more than “two” items, and this can help them understand thateach successive number in the counting sequence is exactly one more than the previous number. |
| ***Monkey Game*.** |
| **Mentally adding one:** | Once children recognize that adding one is related to their number after knowledge, they can construct the number-after rule for adding one: the sum of a number and one is the number after (e.g., seven and one more is the number after seven, or eight). |
| ***Train.*** |
| **Mentally adding larger numbers** | Mentally adding 1 provides a basis for mentally 2 and 3, the near doubles such as 3+4, and decomposition strategies for adding 8 or 9. |
| ***Number Goal (Decomposition) Game***. |

**E.4: Progression for the Meaningful Development of the Subtraction-as-Addition Reasoning (Subtraction) Strategy**

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| **1. Informal knowledge of**  **addition & subtraction as changing an initial collection**  The addition of items to an original collection makes the collection larger; taking items from an original collection make the collection smaller | | |  | **5. Basic formal knowledge of**  **addition & subtraction including a**  **part-whole meaning**  Collections/counting numbers added together make a whole that is larger than either part; subtracting a part from a whole leaves a part that is smaller than the whole | | | |  | | | **7. Fluency with related**  **addition combinations**  Achieves Phase 3  with addition  complements such as  5+3=8 & 3+5=8 | | |
|  |  |  | |  | **** |  | | | |  | |  |  |
| **2. Empirical**  **inversion**  Adding to an original collection or number and then subtracting the same amount or number  (e.g., computes the sum of 5 + 3 is 8—then separately computes that  the difference of 8 – 3 is 5) |  | **3. Undoing**  **concepta**  Addition & subtraction are ***related*** because adding and subtracting the same number of *items* undo each other (e.g., adding 5 to 3 and then subtracting 5 from 3 makes 3 again | |  |  |  | | | | | |  |  |
| **4.**  **Shared-numbers**  **concept**  Related addition and subtraction combinations share the same ***three*** numbers  (e.g., 5+3=8, 3+5=8, 8–3=5, 8–5=3 all involve the numbers  3, 5, 8) |  | **6. Shared-parts-and-whole concept**  An addition-subtraction family of facts share the same whole and parts | |  | | **8. Complement principle**: If Part a + Part b = Whole c, then the Whole c – Part a = Part b |  | **10. Fluent subtraction-as-addition strategy:**  Non-consciously and automatically  use stored triad  (e.g., efficiently retrieve the difference of  8 – 3 = ?  by accessing the 3-5-8  triad stored in long-term memory) |
|  |
| **9. Deliberate subtraction-as-addition strategy**  Consciously use known sum to deduce difference (e.g., 8 – 3 can be solved by asking what? + 3 = 8) |

***Note***. All un-shaded cells are conceptual knowledge. The orange-shaded Cell 2 is an experience that can lead to a concept. The blue-shaded Cell 9 is conscious procedural knowledge. The green-shaded Cell 10 is compiled integrated conceptual procedural knowledge of the complement principle and subtraction strategy.

**a**The undoing concept is a basic and informal understanding that evolves into the formal, explicit, and general knowledge of the inverse principle—ultimately including the ability to summarize the principle algebraically as *a* + *b* – *b* = *a* or *a* – *b* + *b* = *a*.

From: Baroody, A. J. (2015). Curricular approaches to introducing subtraction and fostering fluency with basic differences in grade 1. In R. Bracho (Ed.), The development of number sense: From theory to practice. *Monograph of the Journal of Pensamiento Numérico y Algebraico (Numerical and Algebraic Thought)*. University of Granada.

**APPENDIX F: TIPS ON TEACHING**

**EARLY CHILDHOOD MATHEMATICS EDUCATION COURSES**

**1. Teach mathematics content, mathematical psychology, and mathematics methods in an integrated fashion**. Teach mathematics in a way that models how you want your students to teach their students.

**2. Regularly engage you students in mathematical inquiry.** Use problem solving as a basis of content instruction and assign problems as homework every class. Regularly engage your students in inductive reasoning (discovering regularities), conjecturing, deductive (logical) reasoning, and justifying.

**3**. **Encourage your students to use their informal knowledge to solve problems.** Do not accept formal solutions unless your students can justify it. This will underscore the importance of building on their students’ informal knowledge and justifying solutions. It will also instill confidence in their informal knowledge and the idea that problems can be solved even if you can’t remember the formula.

**4**. **Have your students work in small groups while solving problems in class, solving home work problems, and answering questions on their readings.** Require your students to arrive at a unanimous consensus to underscore the importance of listening carefully to everyone’s viewpoint and clearly justifying one’s solution method and solution.

**5. Resist the temptation to simply give students solutions and instead play the “devil’s advocate”**. Encourage them to think for themselves, devise and justify their solutions, and reach a consensus. Create cognitive conflict by questioning students’ solution strategy or solution and posing alternative solution strategies or solutions. Better yet, encourage each group to share and justify their solution and debate the merits of each solution.

**6. Help students learn informal analogies for mathematical ideas.** Such analogies are useful ways of relating or connecting formal mathematical ideas to students’ informal or everyday knowledge and make it more likely they will understand the formal concepts or procedures, For example, a familiar analogy for the concept of an angle is the “amount of turn” (e.g., a half turn is a “180”).

**7. Have fun**. Play math games that illustrate how such games could be used to help students construct concepts or learn skills. Illustrate how interesting science experiments use math and could be a basis constructing concepts or learning skills.