### 2. Learning Trajectories for Number and Operations

### Basic number and arithmetic concepts and skills are an essential foundation for understanding and learning higher-levels and other aspects of school mathematics. Such knowledge is also critical for everyday functioning in our information- and technology-based society. A comprehensive discussion of the teaching and learning of number and arithmetic concepts and skills from birth to grade 3 is not possible in a short brief (see Appendix A for an overview). Instead, the brief will serve to summarize (a) a developmental trajectory of early (preschool to grade K) development involving number, counting, numerical relations, and basic arithmetic and (b) how a big idea can serve as the basis for primary-grade (grade K to 3) learning across the domains of number and operations.

### a. Early Learning Trajectory Involving Number and Counting, Numerical Relations, and Operations on Number

#### Although it is unclear what preverbal children understand about the “intuitive numbers” (one to three), numerical relations or operations on numbers, what is clear that language in the form of the number words and quantitative terms such as “more” are critical for the development of a verbal-based number concepts and skills. Summarized below and in Appendix A is learning trajectory that includes children’s earliest concepts and skills involving number, counting, numerical relations, and operations on number. With each step in the learning trajectory, the focus initially should be on working with small collections of objects (one to three items) and then gradually moving to progressively larger collections of objects. Indeed, children may start a new step with small numbers before moving to larger numbers with the previous step.

**Step 1. Verbal subitizing**: *Immediately recognizing the total (cardinal value) of a collection without counting and labeling the total with an appropriate number word*. Initially number words may have little or no meaning. For example, 18-month Arianne bounces down the stairs while saying “Two, two, two, two.” Soon though they have a sense that number words have something to do with quantity but this understanding is inexact. For instance, responding to a question about “how many fingers?” and seeing two, three, four, five, and ten fingers held up in turn, Arianne responds “two” to all. In time, children may use “one” and “two” with reliable accuracy, but treat other number words such as “three” as meaning “many” (as label for 3, 4, or more things). Gradually, other number words take on an exact meaning. As the discussion of succeeding steps illustrates, the ability to verbally subitize collections up to about three is a key foundation for other verbal-based number and arithmetic concepts and skills. The development of verbalizing appears to be dependent on experiences with identifying collections with number words. Two key guidelines for fostering this foundational knowledge—suggestions that include number-targeted teacher talk—are:

 *Experiences that involve labeling a wide variety of examples of homogeneous collections and then heterogeneous collections may help children develop verbal subitizing more quickly.* Labeling various examples of single instances “one,” different examples of pairs “two,” and diverse cases of triplets “three” may help children abstract a concept of one, two, and three. This can help them understand that a wide variety of physical characteristics are irrelevant to number concepts and prompt their search for a common attribute (number). Consider, for instance, the *Can You Find?* game in which a parent or teacher might put out a large and a small blue, red, and yellow block and point to the two red blocks and announce “two red blocks.” Follow up questions for a child or children might include “Can you find and give me two blue blocks?” “Can you find the two blocks on this end [point],” “Can you find and give me two big blocks?” This illustrates that the color, location, or size of an object can define what constitutes a particular collection but that they do not apply to examples of a number and thus are not critical (defining) attributes of a number. Once children can reliably recognize homogeneous collections of a particular number, questions about heterogeneous collections (e.g., “How many toys are in your toy box here?”) may help deepen or broaden their understanding of a cardinal number.

 *Explicitly pointing out non-examples of a number may more readily define the boundaries of a number concept.* (a) One way of comparing and contrasting number words is to introduce them in pairs. Adults might help children construct cardinal concepts by first focusing on one and two, next on two and three, and then on three and four. For instance, reliable identification of one and two can serve as the lower boundary for “three” and examples of “four” can help define its upper boundary. (b) A second method is systematically labeling collections as “*n*” and “not *n.*” For example, after labeling two fingers “two,” a parent or teacher could hold up one, three, four, and five fingers, in turn, and label each as “not two.” Contrasting examples and non-examples can be done in the context of a game such as *Number—Not the Number*

***.*** For example, a child can point to all the collections of two s/he can see and then to all the non-examples of two s/he sees (“Point to something that is not two”). For small groups of children, players can take turns pointing out an example and a non-example of a number. In either case, the game can be made more challenging by putting a time limit on the pointing out process or done in small groups, where children take turns identifying an example of number and a non-example. (c) Children’s errors can serve as an opportunity to point out non-examples and provide precise feedback. For instance, if a child misidentifies a picture of three bears as “two,” a parent or preschool teacher could say, “That’s not two bears; it’s *three* bears.” (d) Perhaps less discomfiting and often highly enjoyable, adults can use error-detection games, in which children indicate when an adult or muppet makes a number identification error.

**Step 2. Meaningful object counting**:*Counting a collection using one and only one number word per object to determine the total or cardinal value of a collection.* Children begin to learn the counting sequence (“one, two, three, four…”) relatively early but even verbally counting up to ten may done with little l quantitative understanding or meaning. Even after children have learned to use number words in a one-to-one fashion with pointing to each item in a collection, many exhibit behavior that indicates they do not really understand the purpose of counting (to determine its total). Specifically, they may not understand that such a process is another way of determining the total or cardinality of a collection and that the last counting word used in this process has special significance because this number word not only serves to indicate mark the last item as counted but also represents the total of all the items counted. This understanding is called the cardinality principle (of counting). For instance, they often recount a collection when asked “how many?” A key reason for this is that young children are often taught to do one-for-one object counting by rote. Key guidelines for fostering this foundational knowledge are:

 *Introduce one-for-one object counting in a meaningful manner by modeling the counting process with collections a child can already verbally subitize.* When adults model the one-for-one counting process, they often either emphasize the last number word or repeat it to indicate that it has special significance (indicates the total or cardinal value of the collection). For example, when illustrating counting with a collection of three items, they will often either count “One, two, ***t-h-r-e-e***” or say “One, two, three—see *three*.” If a child can verbally subitize three and already knows there are three (“Yah, there’s three there”), then there is a decent chance the child will understand the emphasis put on the last number word or why its repeated. That is, the child has a reasonable chance of recognizing that one-for-one object is another way of determining the total and learning the cardinality principle.

 *As with all content areas, instruction and practice counting one-for-one should be done in a purposeful manner—that is, within the context of a real or interesting situation.* There are nu­merous everyday opportunities to use and discuss counting collections (e.g., counting the number of children at a table so that the correct number of crackers, milk containers, project items, or instructional materials can be distributed to the group). Dice games, card games and numerous other games involve counting collections either to play the game or keep score. For example, the *Hidden Stars* game (Baroody, 1987) entails showing a child a card with a collections of stars (or other object, shapes, or symbols), asking the child to count the collection, then turning the card over to hide the collection, and finally asking: ”How many stars am I hiding?” This game creates a real reason in the child’s mind to learn or apply the cardinality principle.

**Step 3. Verbal-based magnitude comparisons**: *Using verbal subitizing or one-for-one counting to determine the numerical relation (e.g., “same number” or “more”) between collections first and then using a mental representation of the counting sequence to specify the numerical relation between number words*. Although young children may use verbal subitizing and one-for-one counting to determine the total or cardinality of a collection, it does guarantee they understand the numerical relations among number words. For example, although they may accurately count one-for-one a collection of six and a collection of seven and determine their cardinal value (“six” and “seven,” respectively), they do not necessarily understand that the collection of “seven” is *more than* the collection of “six.” The sub-steps in helping children understand the order-of-magnitude (ordinal relations) of number words are:

 *First, ensure that children understand relational terms as “more” or “fewer.”* This can be done with collections that are obviously differ (collections involving one to three items or with any two collections in which the larger is more twice as large as the smaller. Such experiences can be done in the context of everyday situations, playing math games, or teaching other content, such as reading a children’s story.

 *Next, have children name the larger of two collections they can verbally subitize.* By literally seeing that “two” is more than “one” and “three” is more than “two,”

verbalizing these relations, and relating these relations to the order these number words in the counting sequence (“one, two, three”), children can construct the *increasing-magnitude principle*—that is, realize that the latter a number word comes in the counting sequence, the larger collection it represents.

 *Once children understand the increasing-magnitude principle, encourage them to apply to one-for-one counting with larger collections.* For example, if Jacob has a score of five (represented by five blocks) and Derye has six (represented by six blocks), the children can each count their collection of blocks to see who counted the furthest. If necessary, a teacher can provide scaffolding by counting Jacob blocks and then counting Derye’s blocks and emphasize that Jacob’s count has been surpassed: “Jacob has five, and Derye has ‘one, two, three, four, f-i-v-e, SIX. If further, explicit scaffolding is needed, the teacher can add: “Six is more than five, because six comes after five when we count.”

 *After children have mastered making concrete comparisons using one-to-one object counting and number-after relations, teachers can help them make abstract comparisons of neighboring number words*. First ensure that a child is fluent with number-after relations (e.g., knows that “when we count, after seven comes … eight). In this way, children can then mentally apply the increasing-magnitude principle to any two number neighbors for which they know the number-after relation. This can be practice with a math game, such as *Car Race*, in which a player draws a card and must decide which of two number neighbors is larger in order to move his/her racecar the most spaces on a racetrack.

**Step 4. Once children can quantify collections meaningfully and understands verbal-based number relations, they are ready to solve basic addition and subtraction word problems.** Children construct a basic informal understanding of addition and subtraction by operating on small collections. By seeing one block added to two blocks, for example they can formulate the idea that adding more to a collection makes it larger. Similarly, by seeing one block taken from two, they can construct the idea that taking away items from a collection makes it small. Once children have completed Steps 1 to 3, they are ready to apply their basic ideas to solving word problems. Solving meaningful word problems informally can provide an important basis for learning and solving symbolic expressions such as 3 + 2 or 3 – 2 or symbolic equations such as 3 + 2 = 5 or 3 – 2 = 1.

 *First encourage children to solve nonverbal addition and subtraction problems with small collections they can verbally subitize*. For instance, the *Super Hiding* game involves showing a child an initial collection on a mat, covering the collection, placing additional item(s) next to the covered collection for a moment, and then pushing the additional item(s) under the cover also. With subtraction, item(s) are removed from covered collection, shown for a few moments, and then removed from sight.

 *Next encourage children to solve* *word problems using their own self-invented strategy and sharing their informal strategy with other children.* Young children frequently model the meaning of simple addition and subtraction. This usually takes the form of counting objects or verbal counting

#### b. The Big Idea of Equal Partitioning and Its Relation to Various Aspects of Number and Operations in the Primary Grade

#### Children beginning school are typically well acquainted or are quickly becoming acquainted with the issue of sharing a number of items fairly between two people or perhaps among three or four people. Fair sharing is an informal analogy for the big idea of equal partitioning, and can serve as the conceptual foundation for formal (school) instruction on a variety number and arithmetic concepts and skills and, as illustrated in Section 3, the domain of measurement as well. In this section, special attention is paid to how fair sharing can help children understand key aspects of fractions, because this topic is frequently not taught well by teachers or understood by pupils.

## Division. Whole-number division can be informally viewed as fairly sharing a collection of items among a given number of people. A primary-level teacher can lay the conceptual foundation for the formal instruction on division in the intermediate grades by taking advantage of every day situations that involve sharing (e.g., “Here’s a plate of 8 crackers, how many crackers will each of the four of you get if the crackers are shared fairly?”) or by posing fair-sharing problems (e.g., While reading the *Door Bell Rang* by Pat Hutchins, a teacher can ask: If there are now four children and there are 12 cookies, how many cookies will each child get if the cookies are shared fairly?”). By building on the familiar experience of fair sharing, Children as young as kindergarten can use their familiarity with fair sharing to invent strategies for solving such real or imaginary problems. Children commonly use a “dealing out” strategy—give each person one item until all nothing else can be shared. Later, formal notation for division, such as 7 ÷ 2 = 3 r1, can be introduced meaningfully by relating it to fair sharing: seven cookies shared fairly between two children results in each getting three cookies with one cookie leftover (remaining).

## Even and odd numbers.A useful informal analogy for the concept of *even number* is a collection of items in which *all* the items can be shared fairly by exactly *two* people. An odd number can be informally thought of as collection that cannot be shared fairly in this manner. For example, 6 cookies, but not 7, cookies can be shared fairly between two children and, thus, an even number. With 7 cookies either one child gets an extra cookie or, after six of the cookies are shared fairly, one is left over. Using this fair analogy, children easily determine which numbers in the counting sequence are even and which are odd and conclude that every other number starting with 1 is odd and every number starting with 2 is even. Labeling the odd and even numbers a number list or number line that includes 0 can lead a class to debate whether 0 is odd or even. It is even if you follow the “every other number” pattern (and because no cookies shared by two children yields the same fair, if disappointing share of none). Such fair-sharing experiences can provide a basis for understanding the more formal definition of an even number as “an integer evenly divisible by two and discovering the rules for adding even (e) and odd (o) numbers: e + e = e, e + o = o, and o + o = e. For example, it can help students understand what otherwise may seem like a paradox or strange rule—that *adding two odd numbers results in an even number*. However, if a plate of 3 cookies and a plate of 5 cookies are shared fairly between two children, then each gets 1 cookie from the first plate and 2 cookies from the second plate, and the remaining 2 cookies can then be divvied up fairly to give each a total of 4 cookies—an even number of cookies.

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## Fractions. Relating fractions to the familiar situation of fair sharing can help children understand otherwise mysterious concepts and skills and empower them in solving problems involving fractions.

##  *uotient* *or division meaning of fractions*. Fractions can represent various meanings—one of which is division (e.g., 3/4 = 3 divided by 4). A division meaning of fractions can be viewed informally as fairly sharing a (continuous) quantity such as a length of string, (the area of) a pizza or rectangular cake among a given number of people. For example, 3/4 can be related to a fair-sharing problem, such as: *If three small pizzas* (the numerator 3) are *shared fairly among* (the fraction bar) *four people* (the denominator), *namely Priscilla, Queen, Ramella, and Shifra,* *what is the size of each person’s share?*

###  *-of-a-whole meaning of fractions*. A part-of-a-whole interpretation of the fraction ¾, for instance, indicates 3 parts of a whole divided into four equal parts. Note that the solution to a fair-sharing problem, such as the one in the previous bullet (i.e., determining each person’s share), requires thinking in terms of a part-of-a-whole meaning (i.e., What part of a whole pizza does each of the four girls get as a fair share?; again see Figure 2.1). Relating fractions to such fair-sharing problems has two important advantages: (a) Such problems underscores what many children (and teachers) do not fully appreciate—that a part-of-the-whole meaning of fractions involves the special case where all the parts are equal in size. A common error in identifying fractions is to count the part(s) of interest and the total number of parts and use these to write a fraction—even though the parts are in equal in size. In Figure 2.2, for instance, the child wrote 1/3 because one of three (unequal) parts was shaded. (b) Children’s familiarity with fair-sharing situation enables them to devise or invent their own strategies for solving problems involving fraction. Figure 2.3 illustrates three different strategies a second-grade class devised on their own to solve a problem involving sharing 8 pizzas among 5 people.

##  nt fractions. A fair-analogy can help underscore equivalent fractions. For example, sharing 1 pizza between 2 people results in the same size share as sharing twice as many pizzas (2 pizzas) among twice as many people (4 people) or three as many pizzas (3 pizzas) among three times as many people (6 people). This is important, because it underscores that equivalent fractions are related by multiplication, not addition (i.e., involve multiplicative, not additive, reasoning. A common error in solving an equivalence problem such as 1/2 = 2/☐ is think: “Well one was added to the top number (sic) 1 of the first fraction to make the top number (sic) of the second fraction 2, so I’ll add 1 to it to make the bottom number (sic) of the second fraction—so the answer is 2/3. Note in Figure 2.3 that the three solutions underscore that 1 + 1/2 + 1/10 (Solution A) = 1-3/5 (Solution B) = 8/5 (Solution C).

##  Comparing fractions. Children are often confused about comparing fractions, such as 1/3 and 1/4, and determining which is larger, because they tend to think in terms of whole numbers (i.e., 4 is larger than 3, so 1/4 must be larger than 1/3). However, a fair-sharing analogy provides children with a powerful tool for reasoning about relations between fractions. For instance, makes clear that 1/3 is larger than 1/4. Which results in a larger share of pizza for each person: 1 pizza shared fairly among 3 people or 1 pizza shared fairly among 4 people?

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| **Figure 2.1: How Fair-Sharing Problems Can Illustrate Both a Division and a Part-of-a-Whole Meaning of Fractions and Serve as a Bridge Between Them (Baroody & Coslick, 1998)** | | |
| A. Divide each pizza into four parts (pieces). | 3 pizzas shared by 4 | 5 pizzas shared by 4 |
| B. Divvy up the pieces among the four girls: Priscilla (P), Queen (Q), Ramella (R), & Shifra (S). | Each girl gets one of four equal  shares of each pizza. | Each girl gets one of four equal  shares of each pizza. |
| C. The results of divvying up the pieces. |  |  |
| D. Naming the size of each girl's share relative to a whole pizza. (Note that this involves a part-of-the whole meaning.) |  |  |
| E. Symbolic represen-tation of a share. |  |  |

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| **Figure 2.2: A Common Error in Identifying Part-of-a-Whole Fractions** | |
| Write a common fraction to show what part of  the pie was eaten (the darker portion).  1/3 |  |

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| **Figure 2.3: Three Informal Strategies Devised by Second Graders for Solving**  **Problems That Involve Dividing 8 Pizzas Among 5 People** | |
| **Solution**  **A:**  **Solution**  **B:**  **Solution**  **C:** |  |

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| **Appendix A: A Learning trajectory of early number and arithmetic** | | | |
| **Levels of Learning** | **Development, including How Previous Levels in the Developmental Progression Serve as a Basis** | **Relation of Level to the Common Core State Standards (CCSS) Math Content Standards** | **Example of Instructional Activities for Promoting a Level** |
| ***Verbal subitizing*** of collections of 1 and 2 first, then 3, and—in time—4 to 6:  *Concept: Recognition that number words represent a specific total (cardinal value)* Skill: Reliably and immediately identify small collections with an appropriate number word. | By seeing different examples of a number labeled with a unique number word (e.g., “*two* eyes,” “*two* hands,” “*two* socks,” “*two* shoes,” “*two* cars”) and non-examples labeled with other number words (“take one cookie, not two”), children construct precise, verbal-based cardinal concepts one and two and then progressively larger numbers up to about six (Palmer & Baroody, 2011). Children recognize that number words in general represent a specific number of items about 4 years of age (Sarnecka & Carey, 2008; Sarnecka & Gelman, 2004). | ***Verbal subitizing*** is NOT a grade K CCSS goal. It *should* be because (a) many at-risk kindergartners have not mastered verbal subitizing up to 3 (and such a deficiency is a major handicap), and (b) many kindergartners have not yet mastered conceptual subitizing (e.g., see a set of six as “3 and 3 is 6”). | ***The Number—Not the Number Game.*** *P*layers take turns pointing out an example and a non-example of a number. The game can be made more challenging by putting a time limit on the pointing out process.  **Games involving a die or dice.** |
| ***Meaningful object counting***  *Concepts: Understanding (a) 1-to-1 object counting is another way of determining the total number of items and (b) the principles of how to execute accurate counting.*  Skill: Accurately use 1-word-to-1-object counting to discern the cardinal value of sets. | ***Verbal subitizing*** enables children to understand the principles underlying ***meaningful object counting:*** *stable order, one-to-one,* and *count-cardinality principles*. For example, by watching an adult count a small collection a child can recognize as “three,” s/he can understand why the last number word in the count is emphasized or repeated—it represents the total or how many (the cardinal value of the collection). | **KCC.B.4** (Understand the relationship between numbers and quantities; connect counting to cardinality); and **K.CC.B.5** (Count to answer “how many?” questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects). | ***Hidden Stars*** (Baroody, 1987). The hider [the teacher] shows the player(s) some stars pasted on 5 x 8 card. The player counts the stars. Then the hider covers them up and says, “How many stars am I hiding?” The player tries to tell how many stars the hider is hiding. |
| ***Producing collections***  *Concepts:* ***Cardinal-count principle*** *=a cardinal number specifies when to stop taking or counting a collection of a specified number of objects).*  Skill: Can accurately put pout or count out a specified number of objects up to 5, then 10, and eventually 20. | ***Verbal subitizing*** enables children to put out a specified number of objects. Building on verbal subitizing and the count-cardinality principle can help children understand the cardinal-count principle. For instance, seeing another respond to a request for “three” items by counting out three items, a child can see that the counting-out process stopped when the requested amount (“three”) was reached. | ***Producing collections*** is NOT a grade K CCSS goal but *should* be as a quarter to one-half of kindergartners cannot count out collections up to 20. | ***Animal Spots*** (Wynroth, 1986). On their turn, 1 to 4 players throw a die with 0 to 5 (or 10) dots to determine how many pegs ("spots") they can take for their leopard or giraffe (an animal figure cut out of wood with holes drilled for pegs. After a child counts the number of dots on a die/card, the child takes or counts out the specified number of pegs. |
| ***Concrete ordinal knowledge of number and counting***  *Concept:* ***Increasing magnitude principle*** =recognizing that the counting sequence represent increasingly larger quantities.  Skill: Can use counting to determine the larger/largest of several collections (the collection requiring the longest count). | ***Verbal subitizing*** enables children to see that “2 is *more* than 1” item and that “3 is more than 2” items (understand the term “more” and that numbers have an ***ordinal meaning***, as well as cardinal meaning. ***Verbal subitizing*** and ***ordinal number concept*** permit discovery of the **increasing magnitudeprinciple**. This enables them to use the ***meaningful object counting*** to determine the larger of two collections (e.g., 7 items is more than 6 items because you have to count further to get to *seven* than you do for *six*). | **K.CC.C.6** (Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies). | ***Cards More Than*** (Wynroth, 1986). Played like the card game *War*, except that cards have homogeneous and heterogeneous collections of shapes. Children count the shapes on the card they drew to determine who has more (counts further). |
| ***Number-after knowledge* of the counting sequence**  *Concept: Counting is a breakable chain.*  Skills: (a) Specify the number after another without counting from “one” and (b) counting-on from a specified point in the counting sequence. | Familiarity with the counting sequence enables a child to enter the mental representation of the sequence at any point and ***specify the next number*** instead of always counting from one. | **K.CC.A.2** (Count forward beginning from a given number within the known sequence (instead of having to begin at 1). | ***Dominoes Number After*** (Wynroth, 1986). Played like Dominoes, except that a child must find a domino with the number after an end domino. |
| ***Abstract & precise ordinal knowledge of number and counting.***  Skill: Mental comparisons of close/neighboring number  (number after = more) | The use of the ***increasing magnitude principle*** and ***number-after knowledge*** enables children to determine efficiently and ***mentally compare even close numbers*** such as the larger of two neighboring numbers (e.g., “Which is more 7or 8?”). | This level is the basis for **K.CC.C.7** (Compare two numbers between 1 and 10 presented as written numerals.) | ***Race Game*** (Baroody, 1987).Player is asked which of two number neighbors is more and moves token on a race track the chosen number of times. |
| ***Re-representing the counting sequence as the positive integers*** (*n*, *n*+1, [*n*+1]+1, …)  *Concept:* ***Successor principle*** *=* *each successive number in the count sequence is exactly one more than the previous number*  Skills: (a) Can specify that, e.g., it takes 1 more make 4 into 5 & (b) “4+1 is 5.” | **Verbal subitizing** enables children to see that “two” is exactly one more than “one” items and that “three” is exactly one more than “two” items, and this can help them understand the ***successor principle***. | **K.CC.B.4c** (Understand that each successive number name refers to a quantity that is one larger.) | ***Monkey Successor Game****. Monkey sees a banana on,* e.g., the fifth tree of a series of trees labeled in order from 1 to 10. Monkey swings to get banana but comes up 1 (or 2) tree short. Child asked how many more trees the monkey must go to get from Tree 4 to Tree 5. |